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On Some Problems in the Formulation of Optimum Population Policies when Resources are Depletable

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1. Introduction

In recent years, considerable attention has been focused on the study of optimal depletion patterns of exhaustible resources. The existing models are mostly concerned with the possible mitigating effects of technological progress or capital accumulation in the growth process of economies facing exhaustible resource constraints [see, for example, *Dasgupta*, 1973; *Dasgupta/Heal; Solow; Stiglitz; Ingham/Simmons*, and others]. Population is assumed exogenous to these models, and the concern is with jointly solving the optimal depletion of an exhaustible resource, and optimal investment in augmentable capital goods.

The interrelationship between population policies, and depletion patterns of exhaustible resources has been studied by *Koopmans* [1973, 1974], who poses the problem as a trade-off between the survival time of a fixed population and its consumption rates. This line of analysis has been extended by *Lane* [1975], who allows the population itself to be a control variable, and also allows for a conservationist motive in the optimality exercise. Neither study includes the aspect of capital accumulation offsetting the effect of a (rapidly) depleting resource stock. However the study by *Lane* [1977] establishes a link between the interesting study of Koopmans, and the traditional literature on optimum population, without exhaustible resource constraints, studied by *Meade* [1955], *Dasgupta* [1969], *Lane* [1975], *Pitchford* [1974] and others.

In this paper, we attempt a systematic study of optimum population policies in a model in which capital, labor, and an exhaustible resource produce an output which can be consumed or accumulated as capital. The total stock of the resource is given, and the resource use over the (infinite) planning horizon must not exceed this stock. Population is "freely" controllable, and, so, like *Dasgupta* [1969], we are interested in "first-best solutions". Individual "Utility" is derived from consumption (per capita), and "Welfare" is individual utility times the population at each date. The reason for adopting this Classical Utilitarian view of "Welfare" is that with the alternative Average Utilitarian view, there does not even exist a Pareto-optimal program [see Proposition 3.1]. Furthermore, we follow *Meade* [1955] in assuming that when consumption of an individual is "low", his utility is negative, when it is "high", his utility is positive. We show that this is a necessary condition for the existence of a Pareto-optimal program, with the Classical Utilitarian Welfare function [see Proposition 3.2]. Optimality is then defined in terms of the "maxi-

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misation" of the discounted or undiscounted sums of Welfares, by a suitable version of the "overtaking criterion".

We show that Optimal programs can be characterized in terms of a) the Ramsey Rule of capital accumulation, b) the Meade Rule of population, c) the Hotelling Rule of allocation of an exhaustible resource, and d) the transversality condition that the present value of capital and resource stocks converge to zero, over time [see Theorems 4.1–5.3].

We use this characterization to show that when future welfares are undiscounted, an optimal program does not exist, under a set of quite realistic assumptions [see Theorems 5.1 and 5.3]. This is a somewhat disturbing comment on the Classical Utilitarian view of welfare. We note that similar difficulties are also encountered, when exhaustible resources are not treated explicity [for example, in *Dasgupta*, 1969].

In Theorem 6.1, we show that when future welfares are discounted, an optimal program does exist. We note that the methods of proving the existence of an optimal program, in models where population is exogenous, and exhaustible resources are either absent [see *Gale; Brock; Brock/Gale*] or present [see *Dasgupta/Heal*], cannot be applied to our case. Similarly the methods used in models where population is controllable, but exhaustible resources are absent [see *Dasgupta*] also become inapplicable. Thus, our method of proof is new, although it borrows ideas, at several points, from the above stated "traditional methods".

While Theorem 6.1 might appear to lay at rest questions raised about the appropriateness of the Classical Utilitarian view of welfare [in the sense of Koopmans' "mathematical screening"], Theorem 7.1 raises fresh doubts. Here, we show that when future utilities are discounted, an optimal program must be an "extinction program". That is, it is optimal to have the extinction of the human race in finite time. We note that this result holds, even if there are feasible programs with stationary population, for whom "life is enjoyable" at each date [utility of individuals at various dates are bounded away from zero]. It seems that the Classical Utilitarian view places too small a "penalty" on the extinction of the economy, so that with resources depleting and the future being discounted, it is optimal not to have a "future" at all, beyond a finite time.

2. The Model

Consider an economy with a technology given by a production function, G, from R_+^3 to R_+. The production possibilities consist of capital input, K, exhaustible resource input, D, labor input, L, and current output $Z = G(K, D, L)$ for $(K, D, L) \geqslant 0$.

For simplicity, we will identify "population" with "labor input", at each date, and use the terms interchangeably. Capital will be assumed not to depreciate. Thus *total output*, Y, can be defined as $G(K, D, L) + K$ for $(K, D, L) \geqslant 0$. A *total output function*, F (from R_+^3 to R_+) can be defined by

$$F(K, D, L) = G(K, D, L) + K \qquad \text{for } (K, D, L) \geqslant 0. \tag{2.1}$$

The production function, G, is assumed to satisfy:

(A.1) *G is concave, homogeneous of degree one, and continuous for $(K, D, L) \geqslant 0$; it is continuously differentiable for $(K, D, L) \gg 0$.*

(A.2) *G is non-decreasing in K, D, L for* $(K, D, L) \geqslant 0$; $(G_K, G_D, G_L) \gg 0$ *for* $(K, D, L) \gg 0$.

The initial capital and labor inputs, $\underset{\sim}{K}$ and $\underset{\sim}{L}$, and the initial available stock of the exhaustible resource, $\underset{\sim}{M}$, are considered to be historically given and positive. A feasible program is a sequence $\langle K, D, L, Y, C\rangle = \langle K_t, D_t, L_t, Y_t, C_t\rangle$ satisfying

$$\left.\begin{aligned} &K_0 = \underset{\sim}{K}, L_0 = \underset{\sim}{L}, \sum_{t=0}^{\infty} D_t \leqslant \underset{\sim}{M} \\ &Y_t = F(K_t, D_t, L_t),\ C_t = Y_t - K_{t+1} \text{ for } t \geqslant 0 \\ &(K_t, D_t, L_t, Y_t, C_t) \geqslant 0 \text{ for } t \geqslant 0 \\ &L_t = 0 \text{ implies } L_{t+1} = 0 \text{ for } t \geqslant 1. \end{aligned}\right\} \tag{2.2}$$

Associated with a feasible program $\langle K, D, L, Y, C\rangle$ is a *sequence of resource stocks* $\langle M\rangle = \langle M_t\rangle$, given by

$$M_0 = \underset{\sim}{M},\ M_{t+1} = M_t - D_t \text{ for } t \geqslant 0. \tag{2.3}$$

By (2.2), $M_t \geqslant 0$, and $M_{t+1} \leqslant M_t$ for $t \geqslant 0$.

A feasible program $\langle K, D, L, Y, C\rangle$ is called *positive* if $L_t > 0$ for $t \geqslant 0$. It is *interior* if it is positive and $(K_t, D_t) \gg 0$ for all $t \geqslant 0$. It is *regular interior* if it is interior, and $C_t > 0$ for $t \geqslant 0$.

For a positive program $\langle K, D, L, Y, C\rangle$ we denote, for $t \geqslant 0$,

$$\left.\begin{aligned} &(K_t/L_t) = k_t;\ (D_t/L_t) = d_t \\ &(C_t/L_t) = c_t;\ (Y_t/L_t) = y_t. \end{aligned}\right\} \tag{2.4}$$

Preferences are represented by a *utility function, u*, from R_+ to R. The utility function is assumed to satisfy:

(A.3) *u is strictly increasing for* $c \geqslant 0$.

(A.4) *u is continuous and concave for* $c \geqslant 0$; *it is continuously differentiable for* $c > 0$.

(A.5) $u'(c) \to \infty$ *as* $c \to 0$.

(A.6) *There is* $0 < b < \infty$, *such that* $|u(c)| \leqslant b$ *for* $c \geqslant 0$.[2])

[2]) (A.6) is used only in proving the existence of optimal programs in the discounted case i.e. in Section 6. It may be noted that if there is an optimal program which is interior then from the Meade Rule (4.2 in p. 10) it follows that: $c_t u'(c_t) - u(c_t) = u'(c_t) F_{L_t} > 0$ or $u(c_t)/c_t < u'(c_t)$. Since under (A.7) there exists $\bar{c}$ such that $u(c)/c > u'(c)$ for $c > \bar{c}$ this means $u(c_t) < u(\bar{c})$ for all t. (A.6) guarantees that the utility sums along any feasible path is bounded above. Hence one can use the Cantor Diagonal process to establish existence of a program with largest utility sum as in Lemma 6.3 when the welfare function $W(C, L)$ is continuous. It may be noted that continuity of $W(C, L)$ at $C = 0$ or $L = 0$ (or alternatively defining $W(C, L)$ at $C = 0$ or $L = 0$ ensuring continuity) may be a problem when $u(c)$ is not bounded.

3. On Average and Classical Utilitarian Social Welfare Functions

It has been observed in the literature [see, for example, *Dasgupta*, 1969, p. 295] that if we take the index of social welfare to be the Average Utilitarian one [$V(C, L) = u(C/L)$], and formulate our criterion of optimality in terms of the sum of these welfares then there does not exist an optimal program. Quite apart from the ethical objections to the Average Utilitarian index, this consequence is considered to be a strong reason for rejecting it as a measure of social welfare. We feel that the case against adopting the Average Utilitarian index is further strengthened by showing that, under this valuation, even (interior) Pareto-optimal programs do not exist. We demonstrate this in Proposition 3.1.

In adopting the Classical Utilitarian index of social welfare [$W(C, L) = L\, u(C/L)$], it is assumed in addition that when the consumption rate of an individual is "low", his utility is negative; when it is "high", his utility is positive [see, for example, *Dasgupta*, 1969, p. 296]. We demonstrate (in Proposition 3.2) that a necessary condition for the existence of (interior) Pareto-optimal programs, under the classical Utilitarian valuation, is that the utility function, u, has the above-stated properties.

A *Classical Utilitarian welfare function,* $W(C, L)$, is defined by

$$W(C, L) = L\, u(C/L) \quad \text{for} \quad L > 0; \quad W(C, L) = 0 \quad \text{for} \quad L = 0. \tag{3.1}$$

An *Average Utilitarian welfare function,* $V(C, L)$, is defined by

$$V(C, L) = u(C/L) \quad \text{for} \quad L > 0. \tag{3.2}$$

Note that we leave $V(C, L)$ undefined for $L = 0$, as there is no "natural choice" for its value. The choice of $W(C, L) = 0$ for $L = 0$ makes W a continuous function of (C, L), for $(C, L) \geqslant 0$, under (A.4), (A.6). This is the reason for its choice in (3.1).

Clearly Pareto-Optimality and Optimality can be defined in terms of either of the valuations given by (3.1) and (3.2).

A feasible program $\langle K, D, L, Y, C\rangle$ is called *C-Pareto optimal* if there is no feasible program $\langle K', D', L', Y', C'\rangle$ satisfying $W(C_t', L_t') \geqslant W(C_t, L_t)$ for all $t \geqslant 0$, and $W(C_t', L_t') > W(C_t, L_t)$ for some t. A positive program $\langle K, D, L, Y, C\rangle$ is called *A-Pareto optimal* if there is no positive program $\langle K', D', L', Y', C'\rangle$ satisfying $V(C_t', L_t') \geqslant V(C_t, L_t)$ for all $t \geqslant 0$, and $V(C_t', L_t') > V(C_t, L_t)$ for some t.

To define optimality, we consider a discount factor, δ, where $0 < \delta \leqslant 1$, to be given. A feasible program $\langle K^*, D^*, L^*, Y^*, C^*\rangle$ is *C-optimal* if

$$\limsup_{T\to\infty} \sum_{t=0}^{T} \delta^t \,[W(C_t, L_t) - W(C_t^*, L_t^*)] \leqslant 0 \tag{3.3}$$

for every feasible program $\langle K, D, L, Y, C\rangle$. A positive program $\langle K^*, D^*, L^*, Y^*, C^*\rangle$ is *A-optimal* if

$$\limsup_{T\to\infty} \sum_{t=0}^{T} \delta^t \,[V(C_t, L_t) - V(C_t^*, L_t^*)] \leqslant 0 \tag{3.4}$$

for every positive program $\langle K, D, L, Y, C\rangle$.

Proposition 3.1: Under (A.1)–(A.6), *there is no interior A-Pareto-optimal program.*

Proof: Suppose, on the contrary, that there is an interior program $\langle K, D, L, Y, C\rangle$, which is A-Pareto optimal. Consider the sequence $\langle K', D', L', Y', C'\rangle$ given by: $(K_0', D_0', L_0', Y_0') = (K_0, D_0, L_0, Y_0)$, $C_0' = C_0 + (1/2) K_1$; $(K_t', D_t', L_t', Y_t', C_t') = (1/2)(K_t, D_t, L_t, Y_t, C_t)$ for $t \geq 1$. Clearly $\langle K', D', L', Y', C'\rangle$ is an interior program. Now, $C_0' > C_0$ (since $K_1 > 0$), so $c_0' > c_0$, and $V(C_0', L_0') > V(C_0, L_0)$. Also, $C_t' = (1/2) C_t$, and $L_t' = (1/2) L_t$ for $t \geq 1$. So $c_t' = c_t$, and $V(C_t', L_t') = V(C_t, L_t)$ for $t \geq 1$. Hence, $\langle K, D, L, Y, C\rangle$ is not A-Pareto optimal. This contradiction establishes the Proposition. ||

Remark: It is clear from Proposition 3.1 that there is no interior A-optimal program either, a fact which has been noted in the literature. In view of this, in the rest of the paper, we will be concerned only with the notions of C-Pareto optimality and C-optimality. Also, since there is now no scope for confusion, we will refer to these terms simply as Pareto optimality and optimality respectively.

We now proceed to consider the following additional assumption on u:

(A.7) *There is* $0 < \underset{\sim}{c} < \infty$, *such that* $u(c) < 0$ *for* $0 \leq c < \underset{\sim}{c}$; $u(c) > 0$ *for* $c > \underset{\sim}{c}$; $u(\underset{\sim}{c}) = 0$.

Proposition 3.2: Under (A.1)–(A.6), *if there exists an interior Pareto-optimal program* $\langle K, D, L, Y, C\rangle$, *then the utility function,* u, *satisfies* (A.7).

Proof: Given (A.3), the utility function can be one of three types: (i) $u(c) < 0$ for $c \geq 0$; (ii) $u(c) \geq 0$ for $c \geq 0$; (iii) $u(c^1) < 0$ for some $c^1 \geq 0$, and $u(c^2) > 0$ for some $c^2 \geq 0$. If there is an interior Pareto optimal program $\langle K, D, L, Y, C\rangle$, we will show that cases (i) and (ii) cannot occur.

If case (i) occurs, we construct a sequence $\langle K', D', L', Y', C\rangle$ as follows: $(K_0', D_0', L_0', Y_0') = (K_0, D_0, L_0, Y_0)$, $C_0' = C_0 + (1/2) K_1$; $(K_t', D_t', L_t', Y_t', C_t') = (1/2)(K_t, D_t, L_t, Y_t, C_t)$ for $t \geq 1$. Then, $\langle K', D', L', Y', C'\rangle$ is an interior program. Also, $C_0' = C_0 + (1/2) K_1 > C_0$, so $c_0' > c_0$, and $W(C_0', L_0') > W(C_0, L_0)$. Also, $C_t' = (1/2) C_t$, $L_t' = (1/2) L_t$ for $t \geq 1$. So $W(C_t', L_t') = (1/2) W(C_t, L_t) \geq W(C_t, L_t)$, since $u(c) < 0$ for $c \geq 0$. Hence $\langle K, D, L, Y, C\rangle$ cannot be Pareto-optimal, a contradiction. Thus, case (i) cannot occur.

If case (ii), occurs, then for $c \geq 0$, and $0 \leq \theta \leq 1$, $u(\theta c) = u[\theta c + (1-\theta) 0] \geq \theta u(c) + (1-\theta) u(0) \geq \theta u(c)$, since $u(0) \geq 0$. We construct a sequence $\langle K', D', L', Y', C'\rangle$ as follows: $(K_0', D_0', L_0', Y_0', C_0') = (K_0, D_0, L_0, Y_0, C_0)$; $(K_t', D_t', L_t') = (K_t, D_t, 2L_t)$ for $t \geq 1$, $Y_t' = F(K_t', D_t', L_t')$ and $C_t' = Y_t' - K_{t+1}'$ for $t \geq 0$. Then, $W(C_0', L_0') = W(C_0, L_0)$; also, for $t \geq 1$, $C_t' > C_t$, by (A.2), so $W(C_t', L_t') > L_t' u((1/2) c_t) = 2L_t u((1/2) c_t) \geq 2L_t (1/2) u(c_t) = L_t u(c_t) = W(C_t, L_t)$. Since $\langle K', D', L', Y', C'\rangle$ is clearly an interior program, so $\langle K, D, L, Y, C\rangle$ is not Pareto-optimal, a contradiction. Hence, case (ii) cannot occur.

Thus, case (iii) must occur. Since u is continuous, there is some $0 \leq \underset{\sim}{c} < \infty$, such that $u(\underset{\sim}{c}) = 0$. Since u is increasing, $\underset{\sim}{c} > 0$, and $u(c) < 0$ for $0 < c < \underset{\sim}{c}$; $u(c) > 0$ for $c > \underset{\sim}{c}$. ||

In view of Proposition 3.2, we will assume that (A.7) holds, in the rest of the paper. Note that, under this additional assumption, thure is $\tilde{c} > 0$, satisfying (i) $\underset{\sim}{c} < \tilde{c} < \infty$, and (ii) $u(\tilde{c}) - u'(\tilde{c})\,\tilde{c} = 0$.

4. Characterization of Optimality

In this section, we will provide necessary and sufficient conditions for a positive program to be optimal. This characterization is used in Section 5 to examine the question of existence of an optimal program, when future utilities are undiscounted. It is also used in Section 7 to establish an asymptotic property of optimal programs, when future utilities are discounted.

For our purpose, we will assume that the three types of inputs are essential in production, and that the marginal product of the exhaustible resource is infinite at zero resource input.

(A.8) $G(0, D, L) = G(K, 0, L) = G(K, D, 0) = 0$

For $(K, L) \gg 0$, $G_D(K, D, L) \to \infty$ as $D \to 0$.

Furthermore, following *Mitra* [1978], we assume that the exhaustible resource is "important" in production, in the sense that the share of the resource in current output is bounded away from zero.

(A.9) $\beta \equiv \inf\limits_{(K,D,L) \gg 0} [D\, G_D(K, D, L)/G(K, D, L)] > 0.$

Theorem 4.1: Under (A.1)–(A.9), *if a positive program* $\langle K, D, L, Y, C\rangle$ *is optimal, then*

(i) *it is regular interior*

(ii) $u'(c_t) = \delta u'(c_{t+1})\, F_{K_{t+1}}$ for $t \geqslant 0$ (4.1)

(iii) $[c_t\, u'(c_t) - u(c_t)] = u'(c_t)\, F_{L_t}$ for $t \geqslant 1$ (4.2)

(iv) $[F_{D_{t+1}} / F_{D_t}] = F_{K_{t+1}}$ for $t \geqslant 0$ (4.3)

(v) a) $\lim\limits_{t\to\infty} \delta^t\, u'(c_t)\, K_{t+1} = 0$; b) $\lim\limits_{t\to\infty} M_t = 0.$ (4.4)

Proof: First, we establish that $C_t > 0$ for $t \geqslant 0$. Since $\langle K, D, L, Y, C\rangle$ is optimal, $C_t > 0$ for some t. If $C_t = 0$ for some period, then we can find a period s, such that either a) $C_s = 0$, $C_{s+1} > 0$, or b) $C_s > 0$, $C_{s+1} = 0$. Using (A.5), in either case, $\langle K, D, L, Y, C\rangle$ cannot be optimal, since $(L_s, L_{s+1}) \gg 0$. So, $C_t > 0$ for $t \geqslant 0$. By (A.8), $K_t > 0$ for $t \geqslant 0$.

We claim next, that $D_t > 0$ for $t \geqslant 0$. Clearly, $D_t > 0$ for some t. If $D_t = 0$ for some period, then we can find a period, s, such that either a) $D_s = 0$, $D_{s+1} > 0$, or b) $D_s > 0$, $D_{s+1} = 0$. In either case, $\langle K, D, L, Y, C\rangle$ cannot be optimal, by using (A.8), and $(K_s, L_s) \gg 0$, $(K_{s+1}, L_{s+1}) \gg 0$. Hence, $D_t > 0$ for $t \geqslant 0$. Thus, $\langle K, D, L, Y, C\rangle$ is a regular interior program, which is (i).

For $t \geqslant 0$, the expression

$$L_t\, u\{[F(K_t, D_t, L_t) - K]/L_t\} + \delta L_{t+1}\, u\{[F(K, D_{t+1}, L_{t+1}) - K_{t+2}]/L_{t+1}\}$$

must be maximised at $K = K_{t+1}$, among all $K \geqslant 0$, satisfying $K \leqslant F(K_t, D_t, L_t)$ and $F(K, D_{t+1}, L_{t+1}) \geqslant K_{t+2}$. Using (i),

$$L_t\, u'(c_t)[-1/L_t] + \delta L_{t+1}\, u'(c_{t+1})[F_{k_{t+1}}/L_{t+1}] = 0$$

which yields (4.1) directly.

For $t \geqslant 1$, the expression $L\, u\{[F(K_t, D_t, L) - K_{t+1}]/L\}$ must be maximized at $L = L_t$, among all $L > 0$, satisfying $F(K_t, D_t, L) \geqslant K_{t+1}$. Using (i), we have

$$u(c_t) + L_t\, u'(c_t)\{[L_t F_{L_t} - C_t]/L_t^2\} = 0$$

which yields (4.2) immediately.

For $t \geqslant 0$, the expression

$$L_t\, u\{[F(K_t, D, L_t) - K_{t+1}]/L_t\} +$$
$$+ \delta L_{t+1}\, u\{[F(K_{t+1}, D_t + D_{t+1} - D, L_{t+1}) - K_{t+2}]/L_{t+1}\}$$

must be maximized at $D = D_t$, among all $D \geqslant 0$, satisfying $D \leqslant D_t + D_{t+1}$, $F(K_t, D, L_t) \geqslant K_{t+1}$, and $F(K_{t+1}, D_t + D_{t+1} - D, L_{t+1}) \geqslant K_{t+2}$. Using (i), we get $L_t\, u'(c_t)[F_{D_t}/L_t] + \delta L_{t+1}\, u'(c_{t+1})[-F_{D_{t+1}}/L_{t+1}] = 0$ which yields, on simplification,

$$u'(c_t) F_{D_t} = \delta u'(c_{t+1}) F_{D_{t+1}}. \tag{4.5}$$

Using (4.1) and (4.5) yields (4.3).

Clearly, there is no feasible program, $\langle K', D', L', Y', C'\rangle$ with $L'_t = L_t$ for $t \geqslant 0$, $C'_t \geqslant C_t$ for $t \geqslant 0$, $C'_t > C_t$ for some t. Hence, following the proof of Theorem 4.1 in *Mitra* [1978],

$$\lim_{t\to\infty} M_t = 0 \quad \text{and} \quad \lim_{t\to\infty} [K_{t+1}/F_{D_t}] = 0. \tag{4.6}$$

Note that by using (4.5) repeatedly we have

$$[u'(c_0) F_{D_0}]/F_{D_t} = \delta^t u'(c_t). \tag{4.7}$$

Using (4.6) and (4.7), we obtain (4.4). ||

Remarks: In Theorem 4.1, (4.1) is the well-known "Ramsey-rule" for optimal investment in the capital good. Similarly, (4.2) is the "Meade rule" for an optimum population [see *Meade*, p. 91; or *Dasgupta*, 1969, p. 299]. The marginal condition given by (4.3) is the "Hotelling rule" for optimal depletion of an exhaustible resource [see, for example, *Dasgupta/Heal*, p. 11]. Finally, (4.4) is the transversality condition that the present value of the capital and resource stocks converges to zero as t becomes indefinitely large. It should be noted that (4.1), (4.2), (4.3) are to be expected, as the relevant variables (capital, population and exhaustible resource use) are "freely" and independently controllable. The interesting difference, caused by the presence of exhaustible resources, is in (4.4). In the parallel exercise of optimum population without exhaustible resources, *Dasgupta* [1969, p. 298] notes that (4.1), (4.2) and (4.4) a) are sufficient conditions of optimality. Furthermore, these conditions are necessary when $0 < \delta < 1$. However, if $\delta = 1$ [the discount rate is zero] then (4.4) a) is generally not necessary. In the present exercise, irrespective of the value of δ, (4.4) is a necessary condition of optimality of a positive program.

Theorem 4.2: Under (A.1)–(A.9), *if a regular interior program* $\langle K, D, L, Y, C\rangle$ *satisfies* (4.1)–(4.4), *then there is a price sequence* $\langle p, q, \omega\rangle$, *with* $(p_{t-1}, q_t, \omega_t) \gg 0$ *for* $t \geqslant 0$, *such that*

$$\delta^t W(C_t, L_t) - p_t C_t + \omega_t L_t \geqslant \delta^t W(C, L) - p^t C + \omega_t L \quad \text{for } (C, L) \geqslant 0, t \geqslant 0 \tag{4.8}$$

$$p_t Y_t - p_{t-1} K_t - q_t D_t - \omega_t L_t \geqslant p_t Y - p_{t-1} K - q_t D - \omega_t L \quad \text{for } (K, D, L) \geqslant 0,\ Y = F(K, D, L), t \geqslant 0 \tag{4.9}$$

$$q_t = q_{t+1} \quad \text{for } t \geqslant 0 \tag{4.10}$$

$$\lim_{t\to\infty} [p_{t-1} K_t + q_t M_t] = 0. \tag{4.11}$$

Proof: Define (p_{t-1}, q_t, ω_t) as follows:

$$p_t = \delta^t u'(c_t), \quad q_t = u'(c_0) F_{D_0}, \quad \omega_t = p_t F_{L_t} \quad \text{for } t \geqslant 0 \tag{4.12}$$

$$p_{-1} = p_0 / F_{K_0}.$$

For $L > 0$, $C \geqslant 0$, $W(C, L)$ is a concave differentiable function of (C, L). Also, $(\partial W/\partial C) = u'(c)$ and $(\partial W/\partial L) = u(c) - cu'(c)$. So, for $C \geqslant 0, L > 0$, we have

$$\begin{aligned}\delta^t W(C, L) - \delta^t W(C_t, L_t) &\leqslant \delta^t u'(c_t)(C - C_t) + \delta^t [u(c_t) - c_t u'(c_t)](L - L_t)\\ &= \delta^t u'(c_t)(C - C_t) - \delta^t u'(c_t) F_{L_t}(L - L_t) \quad \text{[using (4.2)]}\\ &= p_t (C - C_t) - \omega_t (L - L_t) \quad \text{[using (4.11)].}\end{aligned}$$

Rearranging terms yields (4.8), for $C \geqslant 0, L > 0$. Note that $\delta^t W(C_t, L_t) = \delta^t u'(c_t) C_t + \delta^t [u(c_t) - c_t u'(c_t)] L_t$ since $W(C, L)$ is homogeneous of degree one for $C \geqslant 0, L > 0$. Hence, $\delta^t W(C_t, L_t) - p_t C_t + \omega_t L_t = 0$. Thus, if $C \geqslant 0$, and $L = 0$, (4.8) is true trivially, since $W(C, L) = 0$. Thus, for $C \geqslant 0, L \geqslant 0$, (4.8) is established.

For $(K, D, L) \gg 0$, we have

$$F(K, D, L) - F(K_t, D_t L_t) \leqslant F_{K_t}(K - K_t) + F_{D_t}(D - D_t) + F_{L_t}(L - L_t).$$

Multiplying through by p_t, and using (4.1), (4.3), (4.12),

$$p_t F(K, D, L) - p_t F(K_t, D_t, L_t) \leqslant p_{t-1} (K - K_t) + q_t (D - D_t) + \omega_t (L - L_t).$$

Rearranging terms gives us (4.9) for $(K, D, L) \geqslant 0$, $Y = F(K, D, L)$. Note that since F is homogeneous of degree one, so $F(K_t, D_t, L_t) = F_{K_t} K_t + F_{D_t} D_t + F_{L_t} L_t$. Hence $p_t F(K_t, D_t, L_t) = p_{t-1} K_t + q_t D_t + \omega_t L_t$. Thus, if $(K, D, L) \geqslant 0$, and $(K,D,L) >> 0$, then by (A.8), $G(K, D, L) = 0$, and (4.9) is trivially true, since $p_t < p_{t-1}$ by (4.1). Thus, for $(K, D, L) \geqslant 0$, $Y = F(K, D, L)$, (4.9) is established.

Finally, using (4.4), and noting from (4.12), that q_t is constant over time, (4.10), (4.11) follow. ||

Remark: Theorem 4.2 provides a competitive price characterisation of an optimal program.

Theorem 4.3: Under (A.1)–(A.9), *if a feasible program* $\langle K, D, L, Y, C\rangle$ *has associated with it a price sequence* $\langle p, q, \omega\rangle$, *with* $(p_{t-1}, q_t, \omega_t) \geqslant 0$ *for* $t \geqslant 0$, *satisfying* (4.8), (4.9), (4.10), (4.11), *then* $\langle K, D, L, Y, C\rangle$ *is optimal.*

Proof: Let $\langle K', D', L', Y', C'\rangle$ be a feasible program. Using (4.8), we write for $t \geqslant 0$

$$\begin{aligned}
\delta^t W(C'_t, L'_t) - \delta^t W(C_t, L_t) &\leqslant p_t (C'_t - C_t) + \omega_t (L_t - L'_t) \\
&= p_t Y'_t - p_t K'_{t+1} - \omega_t L'_t - p_t Y_t + p_t K_{t+1} + \omega_t L_t \\
&= [p_{t-1} K'_t + q_t D'_t + \omega_t L'_t - p_{t-1} K_t - q_t D_t - \omega_t L_t] + \\
&\quad + [p_t Y'_t - p_{t-1} K'_t - q_t D'_t - \omega_t L'_t] - [p_t Y_t - p_{t-1} K_t - q_t D_t - \omega_t L_t] \\
&\quad - p_t K'_{t+1} - \omega_t L'_t + p_t K_{t+1} + \omega_t L_t \\
&\leqslant [p_{t-1} K'_t + q_t D'_t + \omega_t L'_t - p_{t-1} K_t - q_t D_t - \omega_t L_t] \\
&\quad - [p_t K'_{t+1} + \omega_t L'_t - p_t K_{t+1} - \omega_t L_t] \qquad \{\text{by (4.9)}\} \\
&= p_{t-1} (K'_t - K_t) - p_t (K'_{t+1} - K_{t+1}) + q_t (D'_t - D_t).
\end{aligned}$$

Hence, $\sum_{t=0}^{T} \delta^t [W(C'_t, L'_t) - W(C_t, L_t)] \leqslant p_T [K_{T+1} - K'_{T+1}] + q_0 [\sum_{t=0}^{T} D'_t - \sum_{t=0}^{T} D_t]$ by using (4.10). Note that $\sum_{t=0}^{T} D'_t = \underset{\sim}{M} - M'_{T+1}$, and $\sum_{t=0}^{T} D_t = \underset{\sim}{M} - M_{T+1}$, so we have

$$\sum_{t=0}^{T} \delta^t [W(C'_t, L'_t) - W(C_t, L_t)] \leqslant p_T K_{T+1} + q_0 M_{T+1} = p_T K_{T+1} + q_{T+1} M_{T+1}$$

{using (4.10)}. Hence, using (4.11), $\langle K, D, L, Y, C\rangle$ is optimal. ||

5. The Nonexistence of Optimal Programs when Future Welfares are Undiscounted

In this section, we examine the question of existence of an optimal program when future welfares are not discounted. We show that, under one of two alternative additional assumptions, there does not exist an optimal program. One assumption is that there is a feasible program with constant population, which can produce a (current) output sequence bounded away from zero. The other is that the share of capital in current output is bounded away from zero.

Neither of these assumptions is terribly unrealistic. In fact, if these assumptions are *not* satisfied, the model becomes somewhat uninteresting. Specifically, if the first assumption is not satisfied, there does not seem to be too much point in sticking to an infinite-horizon model. If the second assumption is violated, the role of capital accumulation in off setting the exhaustible resource factor is not captured properly, as capital is treated as "unimportant" in production. But the consequence of *either* of these assumptions is that there does not exist an optimal program, which is somewhat disturbing, if one adopts the "mathematical screening" viewpoint of Koopmans.

We note, however, that the result is not totally unexpected, since in exercises on optimum population without exhaustible resources, a similar difficulty is encountered by *Dasgupta* [1969].

Lemma 5.1: Under (A.1)–(A.9), *and* $\delta = 1$, *if* $\langle K, D, L, Y, C\rangle$ *is an optimal program, then*

$$\sum_{t=0}^{T} W(C_t, L_t) \text{ is convergent.} \tag{5.1}$$

Proof: If $L_t = 0$ for some $t = T$, then $L_t = 0$ for $t \geqslant T$, and $W(C_t, L_t) = 0$ for $t \geqslant T$. In this case (5.1) is trivial.

Otherwise $L_t > 0$ for $t \geqslant 0$. In this case $\langle K, D, L, Y, C\rangle$ is a positive program which is optimal. Hence, by Theorems 4.1, 4.2, there is a price sequence $\langle p, q, \omega\rangle$, with $(p_t, q_t, \omega_t) \geqslant 0$ for $t \geqslant 0$, such that (4.8)–(4.11) hold. Using the homogeneity of degree one, of W and F, we then have

$$\begin{aligned} W(C_t, L_t) &= p_t C_t - \omega_t L_t = p_t Y_t - p_t K_{t+1} - \omega_t L_t \\ &= [p_t Y_t - p_{t-1} K_t - \omega_t L_t - q_t D_t] + [p_{t-1} K_t - p_t K_{t+1}] + q_t D_t \\ &= [p_{t-1} K_t - p_t K_{t+1}] + q_t D_t = [p_{t-1} K_t - p_t K_{t+1}] + q_0 D_t. \end{aligned}$$

So

$$\begin{aligned} \sum_{t=0}^{T} W(C_t, L_t) &= [p_{-1} K_0 - p_T K_{T+1}] + q_0 \sum_{t=0}^{T} D_t \\ &= [p_{-1} K_0 - p_T K_{T+1}] + q_{T+1} [\underset{\sim}{M} - M_{T+1}]. \end{aligned}$$

The right-hand side converges by (4.11), as $T \to \infty$, so the left-hand side converges too. In fact, $\sum_{t=0}^{\infty} W(C_t, L_t) = p_{-1} \underset{\sim}{K} + q_0 \underset{\sim}{M}$. This establishes (5.1). ||

Now we consider the following additional assumption:

(A.10) *There is a feasible program* $\langle \bar{K}, \bar{D}, \bar{L}, \bar{Y}, \bar{C}\rangle$ *with* $\bar{L}_t = \underset{\sim}{L}$ *for* $t \geqslant 0$, *and* $\inf_{t \geqslant 0} G(\bar{K}_t, \bar{D}_t, \bar{L}_t) > 0$.

For necessary and sufficient conditions on G, such that (A.10) is satisfied, see *Cass/Mitra* [1979].

Theorem 5.1: Under (A.1)–(A.10), *and* $\delta = 1$, *there does not exist an optimal program.*

Proof: Let $\inf_{t\geq 0} G(K_t, D_t, L_t) \equiv d > 0$. Clearly, $\langle \bar{K}, \bar{D}, \bar{L}, \bar{Y}, \bar{C}\rangle$ is an interior program.

Define $u((\underset{\sim}{c} + \tilde{c})/2) = e$. Then $e > 0$. Let J be a positive integer such that $[J\beta d/\underset{\sim}{L}] \geqslant$ $\geqslant [\underset{\sim}{c} + \tilde{c}]$. Choose $0 < \lambda < 1$, such that $(1/\lambda) \geqslant 2^J$.

Define a sequence $\langle K, D, L, Y, C\rangle$ as follows: $(K_0, L_0, D_0, Y_0) = (\bar{K}_0, \bar{L}_0, \bar{D}_0, \bar{Y}_0)$; $K_t = \lambda \bar{K}_t$, $D_t = \bar{D}_t$, $L_t = \lambda \bar{L}_t = \lambda \underset{\sim}{L}$ for $t \geqslant 1$, $Y_t = F(K_t, D_t, L_t)$ for $t \geqslant 1$; $C_t = Y_t - K_{t+1}$ for $t \geqslant 0$. Clearly, $C_0 > \bar{C}_0 \geqslant 0$. We will show that $C_t > 0$ for $t \geqslant 1$, so $\langle K, D, L, Y, C\rangle$ is a feasible program. We check this fact with the following calculations. For $t \geqslant 1$, $G(K_t, D_t, L_t) = G(\lambda \bar{K}_t, \bar{D}_t, \lambda \bar{L}_t) = \lambda G[\bar{K}_t, (\bar{D}_t/\lambda), \bar{L}_t] \geqslant$
$\geqslant \lambda G(\bar{K}_t, 2^J \bar{D}_t, \bar{L}_t) = \lambda \sum_{j=1}^{J} [G(\bar{K}_t, 2^j \bar{D}_t, \bar{L}_t) - G(\bar{K}_t, 2^{j-1}\bar{D}_t, \bar{L}_t)] + \lambda G(\bar{K}_t, \bar{D}_t, \bar{L}_t)$.
Now for $J \geqslant j \geqslant 1$, we have $G(\bar{K}_t, 2^j \bar{D}_t, \bar{L}_t) - G(\bar{K}_t, 2^{j-1}\bar{D}_t, \bar{L}_t) \geqslant$ $(1/2)\, G_D[\bar{K}_t, 2^j \bar{D}_t, \bar{L}_t]\, 2^j \bar{D}_t \geqslant (1/2)\,\beta G[\bar{K}_t, 2^j \bar{D}_t, \bar{L}_t] \geqslant (1/2)\,\beta d$. Hence, for $t \geqslant 1$, $G(\bar{K}_t, 2^j \bar{D}_t, \bar{L}_t) - G(\bar{K}_t, \bar{D}_t, \bar{L}_t) \geqslant (1/2)\, J\beta d$. Thus, for $t \geqslant 1$, we have

$$
\begin{aligned}
C_t &= G(K_t, D_t, L_t) + K_t - K_{t+1} \\
&= \lambda\, G(\bar{K}_t, (\bar{D}_t/\lambda), \bar{L}_t) + \lambda \bar{K}_t - \lambda \bar{K}_{t+1} \\
&\geqslant \lambda\, G(\bar{K}_t, \bar{D}_t, \bar{L}_t) + (1/2)\, \lambda J \beta d + \lambda \bar{K}_t - \lambda \bar{K}_{t+1} \\
&= \lambda\, \bar{C}_t + (1/2)\, \lambda J \beta d > 0.
\end{aligned}
$$

Since, for $t \geqslant 1$, $C_t \geqslant (1/2)\, \lambda J \beta d$, so $c_t \geqslant [J\beta d/2\underset{\sim}{L}]$, and $u(c_t) \geqslant e$ for $t \geqslant 1$. So $L_t\, u(c_t) \geqslant \lambda \underset{\sim}{L} e$ for $t \geqslant 1$. Thus, as $T \to \infty$, $\sum_{t=0}^{T} W(C_t, L_t) \to \infty$.

If there is an optimal program, $\langle K^*, D^*, L^*, Y^*, C^*\rangle$, then by Lemma 5.1, $\sum_{t=0}^{\infty} W(C_t^*, L_t^*)$ is convergent. But since $\langle K, D, L, Y, C\rangle$ is a feasible program with $\sum_{t=0}^{T} W(C_t, L_t) \to \infty$ as $T \to \infty$, so $\langle K, D, L, Y, C\rangle$ could not be optimal. ||

We now consider an alternative additional assumption:

(A.11) $\alpha \equiv \inf_{(K,D,L) \gg 0} [K\, G_K(K, D, L)/G(K, D, L)] > 0.$

(A.11) states that the share of capital in current output is bounded away from zero. [By (A.9), $\alpha < 1$].

Theorem 5.2: Under (A.1)–(A.9), (A.11), *and* $\delta = 1$, *there does not exist an optimal program.*

Proof: Define $g(K) = G(K, 1, 1)$ for $K \geqslant 1$. Then $g'(K) = G_K(K, 1, 1)$, and $[Kg'(K)/g(K)] = [K\, G_K(K, 1, 1)/G(K, 1, 1)] \geqslant \alpha$. So $[g'(K)/g(K)] \geqslant [\alpha/K]$, and

$(d/dK)\,[\log g\,(K)] \geqslant (d/dK)\,[\log K^\alpha]$. For $x \geqslant 1$, we have $\int_1^x (d/dK)\,[\log g\,(K)]\,dK \geqslant \int_1^x (d/dK)\,[\log K^\alpha]\,dK$. So $\log g\,(x) - \log g\,(1) \geqslant \log x^\alpha - \log 1^\alpha = \log x^\alpha$, and $\log g\,(x) \geqslant \log\,[g\,(1)\,x^\alpha]$. Then $g\,(x) \geqslant g\,(1)\,x^\alpha$ for $x \geqslant 1$, and $G\,(K, 1, 1) \geqslant g\,(1)\,K^\alpha$ for $K \geqslant 1$. Define $\lambda = 1 + [\alpha/2\,(1-\alpha)]$. Choose $0 < \theta < 1$, such that

$$\sum_{t=1}^{\infty} [\theta \underset{\sim}{K}/t^\lambda] \leqslant \frac{1}{2}\,\underset{\sim}{M}.$$

Define a sequence $\langle K, D, L, Y, C\rangle$ as follows: $K_t = \underset{\sim}{K}$ for $t \geqslant 0$; $D_0 = (1/2)\,\underset{\sim}{M}$, $D_t = [\theta \underset{\sim}{K}/t^\lambda]$ for $t \geqslant 1$; $L_0 = \underset{\sim}{L}$, $L_t = [\theta \underset{\sim}{K}/t]$ for $t \geqslant 1$; $C_t = G\,(K_t, D_t, L_t)$ for $t \geqslant 0$; $Y_t = F\,(K_t, D_t, L_t)$ for $t \geqslant 0$. Clearly, $\langle K, D, L, Y, C\rangle$ is a feasible program.

Now, for $t \geqslant 1$, we have $C_t = G\,(K, [\theta \underset{\sim}{K}/t^\lambda], [\theta \underset{\sim}{K}/t]) \geqslant G\,(\theta \underset{\sim}{K}, [\theta \underset{\sim}{K}/t^\lambda], [\theta \underset{\sim}{K}/t])$ $= \theta \underset{\sim}{K}\,G\,(1, [1/t^\lambda], [1/t]) = [\theta \underset{\sim}{K}/t^\lambda]\,G\,(t^\lambda, 1, t^{\lambda-1}) \geqslant [\theta \underset{\sim}{K}/t^\lambda]\,G\,(t^\lambda, 1, 1) \geqslant [\theta \underset{\sim}{K}/t^\lambda]$ $t^{\alpha\lambda} g\,(1) = g\,(1)\,\theta\,\underset{\sim}{K}\,t^{(\alpha\lambda-\lambda)}$. Hence $c_t \geqslant [g\,(1)\,\theta \underset{\sim}{K}\,t^{[(\alpha-1)\lambda+1]}/\theta \underset{\sim}{K}] = g\,(1)\,t^{[1+(\alpha-1)\lambda]}$. By definition of λ, $1 + (\alpha - 1)\,\lambda = (\alpha/2)$. So, $c_t \geqslant g\,(1)\,t^{(\alpha/2)}$ for $t \geqslant 1$. So there is $T_1 < \infty$, such that for $t \geqslant T_1$, $c_t \geqslant (\underset{\sim}{c} + \tilde{c})/2$. Hence for $t \geqslant T_1$ $u\,(c_t) \geqslant$ $\geqslant u\,((\underset{\sim}{c} + \tilde{c})/2) = e$, say [clearly, $e > 0$]. And so, for $t \geqslant T_1$, $W\,(C_t, L_t) \geqslant [\theta \underset{\sim}{K} e/t]$. Hence $\sum_{t=0}^{T} W\,(C_t, L_t) \to \infty$ as $T \to \infty$. If there is an optimal program $\langle K^*, D^*, L^*, Y^*, C^*\rangle$ then $\sum_{t=0}^{\infty} W\,(C_t^*, L_t^*)$ is convergent, by Lemma 5.1. Since $\langle K, D, L, Y, C\rangle$ is a feasible program, and $\sum_{t=0}^{T} W\,(C_t, L_t) \to \infty$ as $T \to \infty$, so $\langle K^*, D^*, L^*, Y^*, C^*\rangle$ cannot be optimal. ||

Remark: If $G\,(K, D, L) = K^\alpha D^\beta L^\gamma$, with $(\alpha, \beta, \gamma) \geqslant 0$, $\alpha + \beta + \gamma = 1$, then (A.10) is satisfied if and only if $\alpha > \beta$. Thus, in the Cobb-Douglas case, (A.10) implies (A.11).

6. The Existence of an Optimal Program when Future Welfares are Discounted

The main result of this section is that, when future welfares are discounted, an optimal program exists.

Readers familiar with the literature on optimum population will recognize that the traditional methods of proving the existence of an optimal program break down, when population is "freely" controllable.[3]) Specifically, given any t, and $M(t) < \infty$, one can find a feasible program with $L_t > M(t)$. Thus, the essential "boundedness" property of the relevant variables, which is exploited heavily in traditional methods [see, for example, *Gale; Brock; Brock/Gale* and others] to arrive at an optimal program as a limit of a convergent (sub)sequence of feasible programs, is not available.

Dasgupta [1969] solves the problem by constructing a particular stationary program (stationary in per-capita magnitudes), and checking that it satisfies the sufficient conditions of optimality of the sort discussed in Section 4. This is in the context of a model without exhaustible resources. When such resources are present, even this clever device is lost, as programs stationary in per-capita magnitudes do not satisfy the appropriate "marginal conditions" of Section 4. [Note that if K_t, D_t, L_t are all growing or decreasing at the same rate, then the marginal products of all three factors must be constant over time; but the "Hotelling Rule" (4.3) demands that the marginal product of the resource be increasing.]

Our method retains the spirit of the traditional (Ramsey) device, though in execution it appears different. We separate feasible programs into two categories: "good" and "bad". "Good" programs are those for which population does not grow "too fast" [in a manner made precise in the definitions below]; "bad" programs are feasible programs which are not "good".

We show that if a feasible program is bad, there is a good program which is "better". There *is* a good program; and, in the class of good programs, there is a "best" program. This is then shown to be an optimal program.

In order to simplify our existence proof (which is still quite elaborate), we assume in this section that the production function is Cobb-Douglas:

(A.12) $G(K, D, L) = K^{\alpha} D^{\beta} L^{\gamma}$, *where* $(\alpha, \beta, \gamma) \geqslant 0$, *and* $(\alpha + \beta + \gamma) = 1$.

We assume throughout, of course, that $0 < \delta < 1$. Given (A.12), we denote the expression $[(\underset{\sim}{K}^{1-\alpha}/\underset{\sim}{L}^{\gamma}) + \underset{\sim}{M}^{\beta}]^{1/1-\alpha}$ by E. Given any feasible program $\langle K, D, L, Y, C\rangle$ we denote

$$A_t = \sum_{s=0}^{t} L_s^{(\gamma/1-\beta)} \qquad \text{for } t \geqslant 0.$$

[3]) We have assumed throughout this paper that population can be controlled arbitrarily that is no bounds are imposed on the rate of growth of population per period. If there are such bounds then the existence question in Section 6 becames easier to handle using standard methods since definite bounds are available on the variable L_t in each period. One would suspect that in this case optimal paths would exist where these constraints are binding in some periods (see the literature on population growth without exhaustible resources where population is arbitrarily variable and where there are constraints on its rate of growth, *Dasgupta* [1969] and *Lane* [1977].

If population control is assumed costly in terms of resources or consumption then it introduces additional elements in the problem which forms a subject of enquiry beyond the scope of the present paper. Since in the case where population is costlessly controlled, along an optimal path in the discounted case, $L_t = 0$ after finite time, it is tempting to conjecture that a similar behaviour would occur when population control is costly with the decline in population taking place at a slower pace possibly happening only in the limit over an infinite horizon. The analysis of the case where population control is costly and is constrained within limits may be an interesting subject of future enquiry.

Lemma 6.1: Under (A.12), *if* $\langle K, D, L, Y, C\rangle$ *is a feasible program, then*

$$K_{t+1} \leqslant E A_t^{(1-\beta)/(1-\alpha)} \qquad \text{for } t \geqslant 0 \tag{6.1}$$

$$C_t \leqslant E A_t^{(1-\beta)/(1-\alpha)} \qquad \text{for } t \geqslant 0. \tag{6.2}$$

Proof: Consider the feasible program $\langle \bar{K}, \bar{D}, \bar{L}, \bar{Y}, \bar{C}\rangle$ given by $\bar{K}_0 = \underset{\sim}{K}$, $\bar{D}_t = D_t$, $\bar{L}_t = L_t$ for $t \geqslant 0$; $\bar{K}_{t+1} = \bar{Y}_t = F(\bar{K}_t, \bar{D}_t, \bar{L}_t)$ for $t \geqslant 0$, and $\bar{C}_t = 0$ for $t \geqslant 0$. Then $\bar{K}_{t+1} \geqslant \bar{K}_t$ for $t \geqslant 0$.

Now, for $t \geqslant 0$, we have $\bar{K}_{t+1} - \bar{K}_t = \bar{K}_t^{\alpha}\bar{D}_t^{\beta}\bar{L}_t^{\gamma}$, so that $\bar{K}_{t+1}^{1-\alpha} - \bar{K}_t^{1-\alpha} \leqslant \bar{D}_t^{\beta}\bar{L}_t^{\gamma} = \bar{D}_t^{\beta}\,[\bar{L}_t^{\gamma/(1-\beta)}]^{1-\beta}$. Using Holder's inequality, we have for $T \geqslant 0$,

$$\bar{K}_{T+1}^{1-\alpha} - \bar{K}_0^{1-\alpha} \leqslant [\sum_{t=0}^{T} \bar{D}_t]^{\beta}\,[\sum_{t=0}^{T} \bar{L}_t^{\gamma/(1-\beta)}]^{1-\beta} \leqslant \underset{\sim}{M}^{\beta}\,\bar{A}_T^{1-\beta}$$

or,

$$\bar{K}_{T+1}^{1-\alpha} \leqslant \bar{K}_0^{1-\alpha} + \underset{\sim}{M}^{\beta}\,\bar{A}_T^{1-\beta} \leqslant E^{(1-\alpha)}\,\bar{A}_T^{(1-\beta)}.$$

So $\bar{K}_{t+1} \leqslant E\,\bar{A}_t^{((1-\beta)/(1-\alpha))}$ for $t \geqslant 0$. Since $K_{t+1} \leqslant \bar{K}_{t+1}$ for $t \geqslant 0$, so (6.1) follows. Also $C_t \leqslant Y_t \leqslant \bar{Y}_t = \bar{K}_{t+1}$ for $t \geqslant 0$, so (6.2) follows. ||

Before proceeding further we introduce some notation. Denote $\beta/(1-\alpha)$ by μ; $(1-\beta)/(1-\alpha)$ by η; $(1-\alpha)/\beta$ by ν. Since we are dealing with the discounted case, we are given $0 < \delta < 1$. Choose $\lambda > 1$, so that $\theta \equiv \lambda^{\nu}\delta < 1$. [Then $\lambda\delta < 1$ also.] Note that $a \equiv \sum_{t=0}^{\infty} ((t+1)^{\eta}/\lambda^{t})$ is convergent. Denote $[2aE/\underset{\sim}{c}]$ by h, and $[h^{\nu}b/(1-\delta)]$ by $\hat{A}$ [where b is given by (A.6)]. Note that $\hat{B} \equiv \sum_{t=0}^{\infty} [\lambda^{\nu}\delta]^{t}$ is convergent. Denote λ^{ν} by π; $\hat{D} = [\underset{\sim}{c}\,u'(\underset{\sim}{c})/2]$. Define $Q = \max[\{2\hat{A}/(1-\theta)\hat{D}\}, 2h^{\nu}, \underset{\sim}{L}]$.

For any feasible program $\langle K, D, L, Y, C\rangle$, we define a sequence $t(n)$ as follows. Let $t(0) = 0$; for $n \geqslant 0$; define $\Omega(n+1) = \{t \geqslant t(n) : L_t > L_{t(n)}\}$, and if $\Omega(n+1)$ is non-empty, $t(n+1) = \min \Omega(n+1)$. If the set $\Omega(n+1)$ is empty for some $n = \bar{n}$, $t(\bar{n}+1) = \infty$, and $t(n)$ is undefined for $n > \bar{n}+1$.

Lemma 6.2: Under (A.3)–(A.6), (A.12), *if* $\langle K, D, L, Y, C\rangle$ *is a feasible program and* $n \geqslant 0$, *such that* $t(n)$ *and* $t(n+1)$ *are defined, then for* $t(n) \leqslant S \leqslant t(n+1) - 1$,

(i) $L_{t(n)}^{\mu} \leqslant h\lambda^{t(n)}$ *implies* $\sum_{t=t(n)}^{S} \delta^{t}\, W(C_t, L_t) \leqslant \hat{A}\,\theta^{t(n)}$

(ii) $L_{t(n)}^{\mu} > h\lambda^{t(n)}$ *implies* $\sum_{t=t(n)}^{S} \delta^{t}\, W(C_t, L_t) \leqslant -\hat{D}\,\delta^{t(n)} L_{t(n)}$.

Proof: To prove (i), note that for $t(n) \leqslant t \leqslant t(n+1) - 1$, $\delta^{t} L_t \leqslant \delta^{[t-t(n)]}\delta^{t(n)} L_{t(n)} \leqslant h^{\nu}\,[\delta\lambda^{\nu}]^{t(n)}\,\delta^{[t-t(n)]}$. Hence, for $t(n) \leqslant S \leqslant t(n+1) - 1$, we have

$$\sum_{t=t(n)}^{S} \delta^{t} W(C_t, L_t) \leqslant h^{\nu}\,[\delta\lambda^{\nu}]^{t(n)}\, b \sum_{t=t(n)}^{S} \delta^{[t-t(n)]} \leqslant \theta^{t(n)} h^{\nu} b/(1-\delta) = \hat{A}\,\theta^{t(n)}.$$

To prove (ii), we write for $t(n) \leq t \leq t(n+1)-1$, $\delta^t L_t u(c_t) = \delta^t L_t [u(c_t) - u(\underset{\sim}{c})]$. So $\delta^t L_t u(c_t) \leq \delta^t L_t u'(\underset{\sim}{c})(c_t - \underset{\sim}{c}) = \delta^t u'(\underset{\sim}{c})[C_t - \underset{\sim}{c} L_t]$. Then for $t(n) \leq S \leq$ $\leq t(n+1)-1$, we have

$$\sum_{t=t(n)}^{S} \delta^t L_t u(c_t) \leq u'(\underset{\sim}{c}) \sum_{t=t(n)}^{S} \delta^t C_t - \underset{\sim}{c}\, u'(\underset{\sim}{c}) \sum_{t=t(n)}^{S} \delta^t L_t . \tag{6.3}$$

Now, for $t(n) \leq t \leq t(n+1)-1$, we have by Lemma 6.1, $\delta^t C_t \leq \delta\, E A_t^{\eta} =$ $\delta^t E [\sum_{i=0}^{t} L_i^{(\gamma/1-\beta)}]^{\eta} \leq \delta^t E [\sum_{i=0}^{t} L_{t(n)}^{(\gamma/1-\beta)}]^{\eta} = \delta^t E (t+1)^{\eta} L_{t(n)}^{(\gamma/1-\alpha)}$. Thus, for $t(n) \leq S \leq$ $\leq t(n+1)-1$, we have

$$\sum_{t=t(n)}^{S} \delta^t C_t \leq E L_{t(n)}^{(\gamma/1-\alpha)} \sum_{t=t(n)}^{S} \delta^t (t+1)^{\eta} \leq E L_{t(n)}^{(\gamma/1-\alpha)} (\delta\lambda)^{t(n)} a.$$

Using this information in (6.3), we have

$$\begin{aligned} \sum_{t=t(n)}^{S} \delta^t L_t u(c_t) &\leq u'(\underset{\sim}{c})\, a E L_{t(n)}^{(\gamma/1-\alpha)} (\delta\lambda)^{t(n)} - \underset{\sim}{c}\, u'(\underset{\sim}{c})\, \delta^{t(n)} L_{t(n)} \\ &= u'(\underset{\sim}{c}) \left[a E L_{t(n)}^{(\gamma/1-\alpha)} (\delta\lambda)^{t(n)} - \frac{1}{2} \underset{\sim}{c}\, \delta^{t(n)} L_{t(n)} \right] - \hat{D}\, \delta^{t(n)} L_{t(n)} \\ &\leq - \hat{D}\, \delta^{t(n)} L_{t(n)} . \qquad \| \end{aligned}$$

We call a feasible program $\langle K, D, L, Y, C\rangle$ *good* if $L_t \leq Q\,\pi^t$ for $t \geq 0$; we call it *bad* if it is not good.

Lemma 6.3: Under (A.3)–(A.6), (A.12), *if a feasible program* $\langle K, D, L, Y, C\rangle$ *is bad, then there is a feasible program* $\langle K', D', L', Y', C'\rangle$ *which is good, such that*

$$\lim_{T\to\infty} \sup \sum_{t=0}^{T} [\delta^t W(C_t, L_t) - \delta^t W(C_t', L_t')] \leq 0. \tag{6.4}$$

Proof: Since $\langle K, D, L, Y, C\rangle$ is bad, there is some t for which $L_t > Q\pi^t$. Let N be the first period this happens. Then, $N > 1$, and $L_{N-1} \leq Q\pi^{N-1} < Q\pi^N < L_N$. So, there is $n > 0$ such that $t(n) = N$. There are now two cases to consider: (i) $t(n+1) = \infty$, (ii) $t(n+1) < \infty$.

In case (i), by Lemma 6.2, $\sum_{t=t(n)}^{S} \delta^t W(C_t, L_t) < 0$ for all $S \geq t(n)$. Define a sequence $\langle K', D', L', Y', C'\rangle$ as follows: $(K_t', D_t', L_t', Y_t', C_t') = (K_t, D_t, L_t, Y_t, C_t)$ for $t < t(n)$, $(K_t', D_t', L_t', Y_t', C_t') = 0$ for $t \geq t(n)$. Then $\langle K', D', L', Y', C'\rangle$ is a feasible program which is good. Also since $\delta^t W(C_t', L_t') = 0$ for $t \geq t(n)$, so (6.4) is satisfied.

In case (ii), by Lemma 6.2, $\sum_{t=t(n)}^{t(n+1)-1} \delta^t W(C_t, L_t) \leq -\hat{D}\, \delta^{t(n)} Q\pi^{t(n)} \leq$ $-2\hat{A}\theta^{t(n)}/(1-\theta)$. Now, for $S \geq t(n+1)$, we have by Lemma 6.2, $\sum_{t=t(n+1)}^{S} \delta^t W(C_t, L_t)$

$\leqslant \sum_{i=n+1}^{\infty} \hat{A}\,\theta^{t(i)} \leqslant \hat{A}\,\theta^{t(n+1)}/(1-\theta)$. So, for $S \geqslant t(n)$, we have $\sum_{t=t(n)}^{S} \delta^t W(C_t, L_t)$ $\leqslant -\hat{A}\,\theta^{t(n)}/(1-\theta)$. Define a sequence $\langle K', D', L', Y', C'\rangle$ as follows: $(K'_t, D'_t, L'_t, Y'_t, C'_t) = (K_t, D_t, L_t, Y_t, C_t)$ for $t < t(n)$, and $(K'_t, D'_t, L'_t, Y'_t, C'_t) = 0$ for $t \geqslant t(n)$. Clearly $\langle K', D', L', Y', C'\rangle$ is a feasible program which is good. Also, since $\delta^t W(C'_t, L'_t) = 0$ for $t \geqslant t(n)$, so (6.4) is satisfied. ||

Lemma 6.4: Under (A.3)–(A.6), (A.12), *there is a good program.*

Proof: Define a sequence $\langle K, D, L, Y, C\rangle$ as follows: $K_t = \underset{\sim}{K}$, $L_t = \underset{\sim}{L}$, $D_t = \underset{\sim}{M}/2^{t+1}$, $Y_t = F(K_t, D_t, L_t)$, $C_t = G(K_t, D_t, L_t)$ for $t \geqslant 0$. Then $\langle K, D, L, Y, C\rangle$ is a feasible program which is good. ||

Lemma 6.5: Under (A.3)–(A.6), (A.12), *there is a good program* $\langle K^*, D^*, L^*, Y^*, C^*\rangle$ *such that*

$$\lim_{T\to\infty} \sup \sum_{t=0}^{T} \delta^t [W(C_t, L_t) - W(C_t^*, L_t^*)] \leqslant 0 \tag{6.5}$$

for every good program $\langle K, D, L, Y, C\rangle$.

Proof: For any good program $\langle K, D, L, Y, C\rangle$, $\sum_{t=0}^{T} \delta^t \mid W(C_t, L_t) \mid \leqslant \sum_{t=0}^{T} (\delta\pi)^t Qb$ $\leqslant [Qb/(1-\delta\pi)] \equiv H$. Hence $\sum_{t=0}^{\infty} \delta^t W(C_t, L_t)$ is absolutely convergent, so $\sum_{t=0}^{\infty} \delta^t W(C_t, L_t)$ is convergent, with $\sum_{t=0}^{\infty} \delta^t W(C_t, L_t) \leqslant H$.

Let $\Lambda = [\sum_{t=0}^{\infty} \delta^t W(C_t, L_t) : \langle K, D, L, Y, C\rangle$ is a good program]. By Lemma 6.4, Λ is non-empty. Also, each element of Λ must be $\leqslant H$. Define $w = \sup \Lambda$; then $w \leqslant H$.

Clearly, there is a sequence $\langle K^i, D^i, L^i, Y^i, C^i\rangle$ of good programs, such that $\sum_{t=0}^{\infty} \delta^t W(C_t^i, L_t^i) \geqslant w - (1/i)$ $[i = 1, 2, \ldots]$. Define $X_0 = \underset{\sim}{K}$; $X_{t+1} = G(X_t, \underset{\sim}{M}, Q\pi^t) + X_t$ for $t \geqslant 0$. Then, if $\langle K, D, L, Y, C\rangle$ is a good program, $(K_t, M_t, L_t, Y_t, C_t) \leqslant$ $\leqslant (X_t, \underset{\sim}{M}, Q\pi^t, X_t, X_t)$ for $t \geqslant 0$. Hence there is a subsequence j of i, such that for each $t \geqslant 0$, $(K_t^j, M_t^j, L_t^j, Y_t^j, C_t^j) \to (K_t^*, M_t^*, L_t^*, Y_t^*, C_t^*)$ as $j \to \infty$. Defining $D_t^* = M_t^* - M_{t+1}^*$, it is easy to check that $\langle K^*, D^*, L^*, Y^*, C^*\rangle$ is a feasible program and it is a good program.

We claim that $\sum_{t=0}^{\infty} \delta^t W(C_t^*, L_t^*) = w$. Otherwise, by definition of w, there is $\epsilon > 0$, such that

$$\sum_{t=0}^{\infty} \delta^t W(C_t^*, L_t^*) \leqslant w - \epsilon.$$

Pick T such that $\sum_{t=T}^{\infty} (\delta\pi)^t Qb < \epsilon/4$. Pick J large enough so that for $j \geqslant J$,

$$| \sum_{t=0}^{T} \delta^t W(C_t^j, L_t^j) - \sum_{t=0}^{T} \delta^t W(C_t^*, L_t^*) | < \epsilon/4.$$

Then, for $j \geqslant J$, we have

$$w - (1/j) \leqslant \sum_{t=0}^{\infty} \delta^t W(C_t^j, L_t^j) = \sum_{t=0}^{T} \delta^t W(C_t^j, L_t^j) + \sum_{t=T}^{\infty} \delta^t W(C_t^j, L_t^j) \leqslant$$

$$\leqslant \sum_{t=0}^{T} \delta^t W(C_t^*, L_t^*) + (\epsilon/4) + (\epsilon/4) = \sum_{t=0}^{\infty} \delta^t W(C_t^*, L_t^*) -$$

$$- \sum_{t=T}^{\infty} \delta^t W(C_t^*, L_t^*) + (\epsilon/2) \leqslant [w - \epsilon] + (\epsilon/4) + (\epsilon/2) = w - (\epsilon/4).$$

So $(1/j) \geqslant (\epsilon/4)$ for $j \geqslant J$, a contradiction. Hence, our claim is established. Then (6.5) follows by the definition of w. ||

Theorem 6.1: Under (A.3)–(A.6), (A.12), *there exists an optimal program.*

Proof: Consider the program $\langle K^*, D^*, L^*, Y^*, C^* \rangle$ whose existence is established in Lemma 6.5. We claim that this is an optimal program.

For, consider any feasible program $\langle K, D, L, Y, C \rangle$. Either this is good or bad. If it is good, then (6.5) holds. If it is bad, then there is a good program $\langle K', D', L', Y', C' \rangle$ such that (6.4) holds. Hence,

$$\limsup_{T \to \infty} \sum_{t=0}^{T} \delta^t [W(C_t, L_t) - W(C_t^*, L_t^*)] \leqslant$$

$$\limsup_{T \to \infty} \sum_{t=0}^{T} \delta^t [W(C_t, L_t) - W(C_t', L_t')] +$$

$$\limsup_{T \to \infty} \sum_{t=0}^{T} \delta^t [W(C_t', L_t') - W(C_t^*, L_t^*)] \leqslant 0$$

by using Lemmas 6.3 and 6.5. Hence, in either case,

$$\limsup_{T \to \infty} \sum_{t=0}^{T} \delta^t [W(C_t, L_t) - W(C_t^*, L_t^*)] \leqslant 0.$$

This establishes our claim. ||

7. Optimality and the Extinction of the Economy

We have shown in Section 5, that under quite realistic assumptions, there does not exist an optimal program when future welfares are undiscounted. While this has been a discomforting result, we have noted that a similar feature is observable in optimal population exercises, even in the absence of exhaustible resource constraints. Furthermore, as in the study of optimum population [without exhaustible resources] by *Dasgupta* [1969], we have been able to establish the existence of an optimal program when future welfares are discounted. Thus, at this stage, the model of production and the Classical Utilitarian objective function may be said to have stood the "mathematical screening" of Koopmans, provided we agree to discount future welfares.

In this section we discover, however, that the discounted case has a disturbing aspect to it also. Namely, optimal programs *must be* extinction programs. This means that it is optimal (in the discounted case) to have a zero population from a certain time period onwards. This result is true independent of whether or not there is a feasible program, with positive stationary population, such that the per-capita consumption at each date generates a utility bounded away from zero. If there is such a feasible program and one finds it optimal to become extinct in finite time, then the Classical Utilitarian objective (with discounting) surely places too small a penalty on the extinction of the human race. This could be viewed as an unsatisfactory aspect of the objective. One might argue that the problem arises because we define $W(C, L) = 0$ rather than $W(C, L) = -\infty$ when $L = 0$. But it is very difficult to justify a discountinuity at $L = 0$ in the objective function, when everywhere else, it is continuous. Furthermore, with this discountinuity we might encounter the problem of non-existence of an optimal program even in the discounted case: notice that if $W(C, L)$ is not continuous everywhere, the existence proof of Section 6 breaks down.

Thus, with the result of this section, we have doubts whether the Classical Utilitarian objective is the appropriate one to use in studying optimum population policies when resources are exhaustible.

Our result should be contrasted with that obtained by *Koopmans* [1974]. An optimal program in the Koopmans exercise is an extinction program, but this is to be expected since there is no aspect of capital accumulation in his model to offset the depletion of resources, and to produce a feasible program, with a utility sequence bounded away from zero. In our model, not only is there the capital accumulation aspect, but capital is smoothly substitutable for the exhaustible resource. Then, with a substitution condition of the type proposed in *Cass/Mitra* [1979], (A.10) will be satisfied. This, in turn, will ensure that there is a feasible program with positive stationary population, and a utility sequence bounded away from zero. However, it will *still* be optimal for the economy to become extinct, according to our result.

We define a feasible program $\langle K, D, L, Y, C\rangle$ to be an *extinction program* if there is an integer $T < \infty$, such that $L_t = 0$ for $t \geqslant T$.

Theorem 7.1: Under (A.1)–(A.9), (A.11), *if* $\langle K, D, L, Y, C\rangle$ *is an optimal program, then it is an extinction program.*

Proof: Suppose on the contrary that $L_t > 0$ for $t \geqslant 0$. Then $\langle K, D, L, Y, C\rangle$ is a positive program, which is optimal. We will now proceed to prove a number of claims, which lead ultimately to a contradiction.

(i) c_t *cannot converge to zero, as* $t \to \infty$. Otherwise, there exists N_1 such that $c_t < \underset{\sim}{c}$ for $t \geqslant N_1$. Construct a sequence $\langle K', D', L', Y', C'\rangle$ as follows: $(K'_t, D'_t, L'_t, Y'_t, C'_t) = (K_t, D_t, L_t, Y_t, C_t)$ for $t < N_1$, $(K'_t, D'_t, L'_t, Y'_t, C'_t) = 0$ for $t \geqslant N_1$. Clearly $\langle K', D', L', Y', C'\rangle$ is feasible. Since $W(C'_t, L'_t) = 0$ for $t \geqslant N_1$, while $W(C_t, L_t) < 0$ for $t \geqslant N_1$, so $\langle K, D, L, Y, C\rangle$ could not be optimal, a contradiction.

(ii) G_{K_t} *cannot converge to zero as* $t \to \infty$. Otherwise, there exists N_2, such that $G_{K_t} \leqslant (1-\delta)/(2\delta)$ for $t \geqslant N_2$. Then, by Theorem 4.1, we have $u'(c_t) = u'(c_{t+1})\,\delta\,(1 + G_{K_{t+1}}) \leqslant u'(c_{t+1})\,\delta\,[1 + (1-\delta)/(2\delta)] = u'(c_{t+1})\{(1+\delta)/2\}$ for $t \geqslant N_2$. Hence $u'(c_t) \to \infty$ as $t \to \infty$, and $c_t \to 0$ as $t \to \infty$. This contradicts (i).

(iii) $G_{D_t} \to \infty$ *as* $t \to \infty$.

Note that by Theorem 4.1, G_{D_t} is increasing with t, so if $G_{D_t} \not\to \infty$ as $t \to \infty$, there is $A < \infty$, such that $G_{D_t} \to A$. By (4.3), we have $G_{D_{T+1}} = G_{D_0} \prod_{t=0}^{T} (1 + G_{K_{t+1}})$. So $\prod_{t=0}^{\infty} (1 + G_{K_{t+1}}) < \infty$, and $\sum_{t=0}^{\infty} G_{K_{t+1}} < \infty$. This implies that $G_{K_{t+1}} \to 0$ as $t \to \infty$, which contradicts (ii).

(iv) $[G_{K_t}/G_{D_t}] \to 0$ *as* $t \to \infty$.

We have $G_{K_t}/G_{D_t} \leqslant F_{K_t}/F_{D_t} = 1/F_{D_{t-1}} = 1/G_{D_{t-1}} \to 0$ as $t \to \infty$ using (4.3) and (iii).

(v) $[D_t/K_t] \to 0$ *as* $t \to \infty$.

By using (A.11), $\alpha \leqslant \{[G_{K_t} K_t]/G(K_t, D_t, L_t)\}/\{[G_{D_t} D_t]/G(K_t, D_t, L_t)\} =$ $= [G_{K_t}/G_{D_t}]\,[K_t/D_t]$. By (iv) $[G_{K_t}/G_{D_t}] \to 0$ as $t \to \infty$, so $K_t/D_t \to \infty$ as $t \to \infty$. That is, $[D_t/K_t] \to 0$ as $t \to \infty$.

(vi) $G_{L_t} \to 0$ *for a subsequence of* t. Otherwise, there is $\theta > 0$, such that $G_{L_t} \geqslant \theta$ for all t. This means that

$$\theta \leqslant G_{L_t} \leqslant \frac{G(K_t, D_t, L_t)}{L_t} = G\left(\frac{K_t}{L_t}, \frac{D_t}{L_t}, 1\right). \tag{7.1}$$

Consequently $[K_t/L_t] \to \infty$ as $t \to \infty$. For if $[K_t/L_t] \leqslant \hat{Q} < \infty$ for a subsequence of t, then $[D_t/L_t] = [D_t/K_t]\,[K_t/L_t] \leqslant [D_t/K_t]\,\hat{Q} \to 0$ for this subsequence, by (v). So $G(K_t/L_t, D_t/L_t, 1) \to 0$ for this subsequence, which violates (7.1). Hence $(K_t/L_t) \to \infty$ as $t \to \infty$, and $[L_t/K_t] \to 0$ as $t \to \infty$.

Now, $G_{K_t} \leqslant \{G(K_t, D_t, L_t)\}/K_t = G(1, D_t/K_t, L_t/K_t)$. Since $[D_t/K_t] \to 0$ as $t \to \infty$, by (v), and $(L_t/K_t) \to 0$ as $t \to \infty$, so $G_{K_t} \to 0$ as $t \to \infty$. This contradicts (ii).

Thus (vi) is established. We now denote $(1+\delta)/2$ by λ; $(1-\delta)/2\delta = e$. Define $S = \{t \geqslant 0 : \delta F_{K_{t+1}} \geqslant \lambda\}$; $S' = \{t \geqslant 0 : \delta F_{K_{t+1}} < \lambda\}$.

(vii) *S and S′ each contain an infinite number of elements.*

If S contains a finite number of elements then there is N_1 such that for $T \geqslant N_1$, $t \in S'$.

So for $t \geqslant N_1$, $u'(c_t) < \lambda u'(c_{t+1})$. So $u'(c_t) \to \infty$ as $t \to \infty$, and $c_t \to 0$ as $t \to \infty$, which contradicts (i).

If S' contains a finite number of elements then there is N_2 such that for $t \geqslant N_2$, $t \in S$. So for $t \geqslant N_2$, $F_{K_{t+1}} \geqslant [\lambda/\delta]$, and $G_{K_{t+1}} \geqslant \{(1+\delta)/2\delta\} - 1 = (1-\delta)/2\delta = e$. So $e \leqslant G_{K_{t+1}} \leqslant \{G(K_{t+1}, D_{t+1}, L_{t+1})\}/K_{t+1} = G(1, D_{t+1}/K_{t+1}, L_{t+1}/K_{t+1})$. By (v), $[L_{t+1}/K_{t+1}] \to \infty$ as $t \to \infty$; so, $(K_{t+1}/L_{t+1}) \to 0$ as $t \to \infty$. Now, $(D_t/L_t) = (D_t/K_t)(K_t/L_t) \to 0$ as $t \to \infty$, since $(K_t/L_t) \to 0$ as $t \to \infty$, and (v) holds. Also, $(K_t/L_t) \to 0$ as $t \to \infty$, so $G(K_t/L_t, D_t/L_t, 1) \to 0$ as $t \to \infty$. Hence, $c_{t+1} \leqslant G(K_{t+1}/L_{t+1}, D_{t+1}/L_{t+1}, 1) + K_{t+1}/L_{t+1} \to 0$ as $t \to \infty$. This contradicts (i). Thus (vii) is established.

Choose $\epsilon > 0$, such that $G[1, \epsilon, (4/\underset{\sim}{c})] \leqslant e$, and $G[(\underset{\sim}{c}/4), (\epsilon\, \underset{\sim}{c}/4), 1] \leqslant (\underset{\sim}{c}/4)$. Choose $\tilde{N}$ such that for $t \geqslant \tilde{N}$, $(D_t/K_t) < \epsilon$.

(viii) *If* $t > \tilde{N}$, $t \in S$, *then* $c_{t+1} \leqslant [\underset{\sim}{c}/2]$.

For $t > \tilde{N}$, $t \in S$, $\delta F_{K_{t+1}} \geqslant \lambda$, so $G_{K_{t+1}} \geqslant e$. So, $e \leqslant G_{K_{t+1}} \leqslant G(1, D_{t+1}/K_{t+1}, L_{t+1}/K_{t+1}) \leqslant G(1, \epsilon, L_{t+1}/K_{t+1})$, and $(L_{t+1}/K_{t+1}) \geqslant [4/\underset{\sim}{c}]$. Hence, $[K_{t+1}/L_{t+1}] \leqslant [\underset{\sim}{c}/4]$, and $c_{t+1} \leqslant \{G(K_{t+1}, D_{t+1}, L_{t+1}) + K_{t+1}\}/L_{t+1} \leqslant G(\underset{\sim}{c}/4, (\epsilon \underset{\sim}{c})/4, 1) + \underset{\sim}{c}/4 \leqslant (\underset{\sim}{c}/2)$. Choose $\bar{N} > \tilde{N}$, such that $\bar{N} \in S$.

(ix) *If* $t \geqslant \bar{N}$, *then* $c_{t+1} \leqslant [\underset{\sim}{c}/2]$.

Suppose, on the contrary, there is some $t \geqslant \bar{N}$, such that $c_{t+1} > [\underset{\sim}{c}/2]$. Consider $t = \tau$ to be the first period this happens. Then τ is not in S, by (viii). So, $\tau \in S'$; also, $\bar{N} \in S$, so $\tau > \bar{N}$. Now,

$$u'(c_\tau) = \delta F_{K_{\tau+1}} u'(c_{\tau+1}) < \lambda u'(c_{\tau+1}) < u'(c_{\tau+1}).$$

So $c_\tau > c_{\tau+1} > [\underset{\sim}{c}/2]$. But since $\tau - 1 \geqslant \bar{N}$, and $c_\tau > (\underset{\sim}{c}/2)$, so τ is not the first period $(\geqslant \bar{N})$, for which $c_{\tau+1} > [\underset{\sim}{c}/2]$, a contradiction.

(x) $\langle K, D, L, Y, C \rangle$ *is not optimal.*

By (ix), $W(C_t, L_t) < 0$ for $t \geqslant \bar{N} + 1$. Construct a sequence $\langle K', D', L', Y', C' \rangle$ as follows: $(K_t', D_t', L_t', Y_t', C_t') = (K_t, D_t, L_t, Y_t, C_t)$ for $t < \bar{N} + 1$; $(K_t', D_t', L_t', Y_t', C_t') = 0$ for $t \geqslant \bar{N} + 1$. Since $W(C_t', L_t') = 0$ for $t \geqslant \bar{N} + 1$, so $\langle K, D, L, Y, C \rangle$ is not optimal.

By (x), $\langle K, D, L, Y, C \rangle$ must be an extinction program. ||

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